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CLOSED CONFORMAL VECTOR FIELDS

SHÛKICHI TANNO & WALDEMAR C. WEBER

1. Introduction

Let M be a connected m-dimensional Riemannian manifold with metric g. A vector field X on M is conformal if and only if

$$L_X g = -(2/m)\delta\xi \cdot g\,, \tag{1.1}$$

where L_X denotes the Lie derivation with respect to X, $\xi = g(X, \cdot)$ is the covariant form of X with respect to g, and $\delta\xi$ is the corresponding codifferential form of ξ. Let d be the exterior differential operator, and Q the Ricci operator which is defined on the 1-form ξ by $Q\xi = R_1(X, \cdot)$, R_1 being the Ricci tensor.

If M is a connected 2-dimensional Riemannian manifold with constant scalar curvature $S > 0$, and M admits a conformal non-Killing vector field, then M is globally isometric with a sphere (cf. [1]).

Next let M be a connected, compact Riemannian manifold of dimension $m > 2$ with constant scalar curvature $S > 0$. Then M is globally isometric with a sphere, if M admits a conformal non-Killing vector field X and any one of the following conditions is satisfied:

(a) *M is an Einstein space* [6], [8],

(b) *the Ricci tensor is parallel* [5],

(c) *trace Q^2 is constant* [4],

(d) *$Qd\delta\xi = kd\delta\xi$ for some constant k* [9],

(e) *$Qd\delta C_0^*(M) \subset d\delta C_0^*(M)$, where $C_0^*(M)$ denotes the space of covariant forms of all conformal vector fields on M* [1],

(f) *ξ is an exact form* [4].

(a) and (e) have been proved independently. Conditions (d) and (e) are related to (f). If ξ is exact, then ξ vanishes at some point of M. The first theorem of this note is a generalization of (f).

Theorem 1. *Let M be a connected compact Riemannian manifold with the constant scalar curvature $S > 0$. Then M is globally isometric with a sphere if M admits a closed conformal vector field X which satisfies one of the conditions:*

(i) *the harmonic part h of ξ vanishes at some point of M,*

(ii) *X vanishes at some point of M,*

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(iii) $R_1(X, X) = (S/m)g(X, X)$ *holds at some point of* M,

(iv) $R_1(X, X)$ *is non-negative.*

Corollary 1. *Let M be connected, compact and orientable Riemannian manifold with positive constant scalar curvature. Assume that the Euler number is not equal to zero* (m: even) *and that M admits a closed conformal vector field. Then M is globally isometric with a sphere.*

Remark. In Theorem 1, any closed conformal vector field satisfies (i), if M has one of the properties:

(v) Q is non-singular at each point of M,

(vi) for any function f, we have $C_0(M, g) \neq I_0(M, e^f g)$, where $C_0(M, \cdot)$ or $I_0(M, \cdot)$ denotes the identity component of the group of conformal transformations or isometries, respectively.

In §4 we consider sufficient conditions for M to admit a closed or exact conformal vector field. We denote by $\Lambda^r(M)$ the space of all r-forms on M and by $H^1(M)$ the space of all harmonic 1-forms on M.

Theorem 2. *Suppose that a compact orientable Riemannian manifold M admits a conformal non-Killing vector field. Then M admits closed conformal vector field if any one of the following conditions is satisfied*:

(a) $Q\delta\Lambda^2(M) \subset \delta\Lambda^2(M)$,

(b) *for any* $w \in \Lambda^1(M)$ *such that* $dw = 0$, *we have* $dQw = 0$,

(c) $dQd\Lambda^0(M) = (0)$ *and* $dQH^1(M) = (0)$,

(d) $\delta Q\delta\Lambda^2(M) = (0)$ *and* $dQH^1(M) = (0)$,

(e) $dQd\delta C_0^*(M) = (0)$, $dQH^1(M) = (0)$ *and S is constant.*

Theorem 3. *Suppose that a compact orientable Riemannian manifold M admits a conformal non-Killing vector field. Then M admits a conformal vector field whose covariant form is exact if any one of the following conditions is satisfied*:

(f) $Qd\Lambda^0(M) \subset d\Lambda^0(M)$,

(g) *for any* $w \in \Lambda^1(M)$ *such that* $\delta w = 0$, *we have* $dQw = 0$,

(h) $dQd\Lambda^0(M) = (0)$ *and* $dQH^1(M) = (0)$,

(i) $\delta Q\delta\Lambda^2(M) = (0)$ *and* $\delta QH^1(M) = (0)$,

(j) $dQd\delta C_0^*(M) = (0)$, $\delta QH^1(M) = (0)$ *and S is constant.*

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2. Preliminaries

Let M be a compact orientable Riemannian manifold. By Yano's theorem [7] a vector field X with the covariant form ξ is a Killing vector field if and only if

$$\delta\xi = 0\,, \quad \langle \Delta\xi - 2Q\xi, \xi \rangle = 0\,, \tag{2.1}$$

where $\Delta = d\delta + \delta d$ is the Laplace-Beltrami operator, and $<\ ,\ >$ denotes the global inner product. Subsequently Lichnerowicz [3] has shown that a necessary and sufficient condition for X to be a conformal vector field is

$$\Delta\xi + (1 - 2/m)d\delta\xi - 2Q\xi = 0\,, \tag{2.2}$$

or

$$\langle \Delta\xi + (2 - 2/m)d\delta\xi - 2Q\xi, \xi\rangle = 0\,. \tag{2.2$'$}$$

Now we assume that the scalar curvature S is constant. We can assume also that S is positive (cf. [3]). The relation $L_X S = 0$ for a conformal vector field X is equivalent to

$$\Delta\delta\xi = (m - 1)^{-1}S\delta\xi\,. \tag{2.3}$$

Let h be the harmonic part of the covariant form ξ of X. Then by the Hodge-de Rham decomposition theorem we have

$$\xi = d\delta\eta + \delta d\eta + h \tag{2.4}$$

for some 1-form η, and substitution of $\delta\xi = \delta d\delta\eta$ into (2.3) gives $\delta(d\Delta\delta\eta - (m-1)^{-1} S\, d\delta\eta) = 0$. Therefore we get $\Delta d\delta\eta = (m-1)^{-1} S\, d\delta\eta$. By $\Delta d\delta\eta = d\delta\xi$, we have

$$d\delta\eta = (m - 1)S^{-1}d\delta\xi\,. \tag{2.5}$$

3. Proof of Theorem 1

To prove Theorem 1 we can assume that $m > 2$ and that M is orientable (cf. [1]). Let X be a closed conformal vector field on M. Then by (2.2) we have

$$\Delta\xi = d\delta\xi = m(m - 1)^{-1}Q\xi\,. \tag{3.1}$$

Since $d\xi = 0$, by (2.4) and (2.5) the harmonic part h of ξ is

$$h = \xi - (m - 1)S^{-1}d\delta\xi\,. \tag{3.2}$$

Operating L_X to $\xi_i = g_{ij}X^j$ gives

$$d(g(X, X)) = L_X\xi = -2m^{-1}\delta\xi\cdot\xi\,. \tag{3.3}$$

Since ξ is closed we have $d\delta\xi\wedge\xi = 0$. Now let M_0 be the set of points where ξ vanishes, and let $M^* = M - M_0$. Then we get a C^∞-function A^* on M^* such that

$$d\delta\xi = A^*\xi \tag{3.4}$$

on M^*. Similarly, we have a C^∞-function B^* on M^* such that

$$dA^* = B^*\xi \tag{3.5}$$

on M^*. By (3.1) we have

$$Q\xi = (m-1)m^{-1}A^*\xi \,. \tag{3.6}$$

Lemma. *The function A^* on M^* is extendable to a continuous function A on M, and*

(A-1) *if $A = (m-1)^{-1}S$ at a point of M^*, then it holds on M,*

(A-2) *if M^* has no point where $A = (m-1)^{-1}S$ holds, then $A < (m-1)^{-1}S$ holds on M, A takes negative values somewhere on M, and M_0 is empty.*

Proof. First we show that $(\delta\xi)_P \neq 0$ for any point P of M_0 when $M \neq M^*$. In fact, if $(\delta\xi)_P = 0$, then (1.1) shows that $(D_j\xi_i)_P = 0$ since ξ is closed, where D is the Riemannian connection by G. By (3.1) we have $(d\delta\xi)_P = 0$. Namely, we have $\xi_P = (D\xi)_P = (\delta\xi)_P = (d\delta\xi)_P = 0$. However, by the differential equations satisfied by conformal vector fields ξ must vanish on M. Therefore we have $(\delta\xi)_P \neq 0$ for any point P of M_0. Then by (1.1) we have $(D_Y\xi)_P \neq 0$ for any non-zero tangent vector Y at P. This means that P is an isolated point. By (3.6) and the continuity of eigenvalues of Q we can extend A^* on M^* to a continuous function A on M. We operate δ to (3.4) and get

$$\delta\Delta\xi = -g(dA^*, \xi) + A^*\delta\xi \tag{3.7}$$

on M^*. By (2.3) and (3.5) we have

$$(m-1)^{-1}S\delta\xi = -B^*g(\xi, \xi) + A^*\delta\xi \tag{3.8}$$

on M^*. We solve (3.8) for B^* and substitute into (3.5). Then using (3.3) we have a differential equation

$$d(A^* - (m-1)^{-1}S) = -2^{-1}m(A^* - (m-1)^{-1}S)d(\log g(X, X)) \tag{3.9}$$

on M^*. Thus, if $A = (m-1)^{-1}S$ at one point of M^*, then $A = (m-1)^{-1}S$ on M. It is known that $\langle Qh, h\rangle \leq 0$ holds for any harmonic 1-form, and so by (3.2) and (3.6) we have

$$\langle (1-(m-1)S^{-1}A)^2Q\xi, \xi\rangle = (m-1)m^{-1}\langle (1-(m-1)S^{-1}A)^2\xi, A\xi\rangle \leq 0\,.$$

This shows that if $A \neq (m-1)^{-1}S$ anywhere on M^* then A takes a negative value at some point on M^*, and hence $(m-1)^{-1}S > A$ on M^*. Then we can solve (3.9) for A as follows:

$$A = (m-1)^{-1}S - e^c(g(X, X))^{-m/2} \tag{3.10}$$

on M^*, where c is a contant. (3.10) means that A can not be continuous on M unless M_0 is empty.

Proof of Theorem 1. In each case we show that the harmonic part of ξ vanishes, and this is equivalent to the relation $A = (m-1)^{-1}S$ by (3.2) and (3.4). (i) follows from the Lemma (A-1). (ii) follows from (A-2) and (A-1) of the Lemma, since M_0 is not empty. (iii) and (iv) follow also from the Lemma.

q.e.d.

Condition (v) of the Remark in the introduction implies that the function A^* does not take value zero. By relations

$$0 = \langle \Delta h, \xi\rangle = \langle h, \Delta\xi\rangle = \langle (1-(m-1)S^{-1}A)A\xi, \xi\rangle\ ,$$

A can not be negative. Then we apply our Lemma.

Let $\xi \in C_0^*(M)$ be closed. If M_0 is non-empty, then ξ is exact. Assume condition (vi) and that M_0 is empty. Set $e^f = (g(X, X))^{-1}$, and define g' by $g' = e^f g$. Then we have $L_X g' = 0$. Next define ξ' by $\xi' = g'(X, \cdot) = (g(X, X))^{-1}\xi$. By (3.3) ξ' is also closed. Therefore ξ' must be a parallel vector field with respect to g', and we have $C_0(M, g') = I_0(M, g')$ (cf. [2]). Since $C_0(M, g') = C_0(M, g)$, this contradicts the assumption.

4. Proofs of Theorems 2 and 3

It may be easily verified that (a), (b), (c) and (d) are equivalent, and (f), (j), (h) and (i) are equivalent. Assuming (c), we show that M admits a closed conformal vector field. Let (2.4) be the decomposition of ξ. Then by (2.2) we have

$$\begin{aligned} 0 &= \langle \Delta\xi + (1-2m^{-1})d\delta\xi - 2Q\xi, \delta d\eta\rangle \\ &= \langle \Delta d\delta\eta + (1-2m^{-1})d\delta d\delta\eta - 2Qd\delta\eta \\ &\qquad - 2Qh + \Delta\delta d\eta - 2Q\delta d\eta, \delta d\eta\rangle\ . \end{aligned}$$

By (c) we have

$$\langle \Delta\delta d\eta - 2Q\delta d\eta, \delta d\eta\rangle = 0\ . \tag{4.1}$$

By (2.1) $\delta d\eta$ defines a Killing vector field, and so $d\delta\eta + h$ is a closed conformal vector field. If S is constant, then $2Qd\delta\eta \in Qd\delta C_0^*(M)$ by (2.5). Thus (e) implies (4.1). This proves Theorem 2. To prove Theorem 3, assume (i). Then by (2.2) we have

$$\begin{aligned} &\langle \Delta d\delta\eta + (1-2m^{-1})d\delta d\delta\eta - 2Qd\delta\eta - 2Qh \\ &\qquad + \Delta\delta d\eta - 2Q\delta d\eta, d\delta\eta\rangle = 0\ . \end{aligned}$$

By (i) we get

$$\langle \Delta d\delta\eta + (1 - 2m^{-1})d\delta d\delta\eta - 2Qd\delta\eta, d\delta\eta \rangle = 0 .$$

Hence $d\delta\eta$ defines a conformal vector field by (2.2)′, which is exact. Case (j) is similar to (e).

Added in proof. Another proof for Theorem 1 appears in W. C. Weber & S. I. Goldberg, *Conformal deformations of Riemannian manifolds*, Queen's Papers in Pure and Applied Mathematics, No. 16, Queen's University, Kingston, Ontario, Canada, 1969.

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TÔHOKU UNIVERSITY
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